Airy Ideals and Witten's Conjecture

Kenny Moc
Supervised by Vincent Bouchard

Abstract

While working on 2-dimensional quantum gravity in 1991, Witten conjectured that two different models of gravity must have the same partition function. This paper explores the concept of Airy Ideals, also known as Airy Structures, within the context of this conjecture. The mathematical underpinnings of Airy Ideals are built from the ground up, culminating in the introduction of Airy Ideals. Furthermore, an important theorem for identifying Airy Ideals is introduced, accompanied by a restated proof. Lastly, we expand on this theorem by weakening some required conditions, with some proofs novel to this paper.

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1 Introduction

1.1 Motivation

This paper explores the topic of Airy Ideals (or equivalently, Airy Structures) as it relates to Witten's Conjecture. Airy Ideals are a certain class of sets of algebraic objects that come with special properties. They began as an algebraic reformulation of topological recursion. However, they can also be understood independently of this origin. The context for our discussion originates in the 1980s, as various approaches emerged as models for two-dimensional quantum gravity. During his work on the subject in 1991, Witten proposed a conjecture asserting the equivalence of two specific models of 2D quantum gravity. Specifically, he suggested that the model based on "intersection theory on moduli spaces of curves" should be related to KdV integrable systems, known for their applications in shallow waves, through their partition functions [1][2]. This statement was highly nontrivial, as it is not immediately clear how these separate fields must be related. Witten's conjecture was later proven by Kontsevich in 1992 [3].

Interestingly, the two models could also be related to each other through the Virasoro constraint. The partition function of the models are related to the Virasoro algebra, as it is uniquely annihilated by its generators [4]. This result can be generalized to encompass other partition functions that are annihilated by distinct sets of differential operators, which we term Airy Ideals.

1.2 Relation to Topological Recursion

The history of Airy Ideals began as an algebraic reformulation of topological recursion [5]. **Topological recursion** is mostly a geometric notion that gives a recursive definition of invariants associated to a spectral curve. Notably, the formalism of topological recursion can be decoupled from spectral curves in general by an algebraic reformulation, giving way to Airy ideals [5]. As an aside, Witten's conjecture can be proven by relating intersection theory on moduli spaces to Hurwitz numbers [6], and the efficient computation of Hurwitz numbers is facilitated by topological recursion [7] [8] [9]. As topological recursion is outside the scope of this paper, further reading on the topic and how it arose from the study of large random matrices can be found in [10].

2 Mathematical Preliminaries

In this paper, we study Airy structures as an ideal (satisfying some conditions) living in a certain completed Weyl algebra. We make this notion rigorous by first defining Rees algebras.

2.1 Rees algebras

Let $\mathcal{D}$ be an associative algebra over $\mathbb{C}$, and $\mathcal{M}$ a left $\mathcal{D}$-module.

Definition 2.1. We define an **exhaustive ascending filtration** on $\mathcal{D}$ to be an increasing sequence of subspaces $\mathcal{F}_i\mathcal{D} \subseteq \mathcal{D}$, for $i \in \mathbb{N}$:

$$0 \subseteq \mathcal{F}_0\mathcal{D} \subseteq \mathcal{F}_1\mathcal{D} \subseteq ... \subseteq \mathcal{D} \tag{2.1}$$

with the additional two properties:

$$\bigcup_{i\in\mathbb{N}} \mathcal{F}_i\mathcal{D} = \mathcal{D} \text{ and } \mathcal{F}_k\mathcal{D} \cdot \mathcal{F}_j\mathcal{D} \subseteq \mathcal{F}_{k+j}\mathcal{D} \tag{2.2}$$

for each $k, j \in \mathbb{N}$. We call any algebra with such a filtration a **filtered algebra**.

A similar definition can be applied to modules:

Definition 2.2. Let $\mathcal{M}$ be a left $\mathcal{D}$-module. An **exhaustive ascending filtration** on $\mathcal{M}$ is an increasing sequence of subspaces $\mathcal{F}_i\mathcal{M} \subseteq \mathcal{M}$, for $i \in \mathbb{N}$:

$$0 \subseteq \mathcal{F}_0\mathcal{M} \subseteq \mathcal{F}_1\mathcal{M} \subseteq ... \subseteq \mathcal{M} \tag{2.3}$$

with the additional two properties:

$$\bigcup_{i\in\mathbb{N}} \mathcal{F}_i\mathcal{M} = \mathcal{M} \text{ and } \mathcal{F}_k\mathcal{M} \cdot \mathcal{F}_j\mathcal{M} \subseteq \mathcal{F}_{k+j}\mathcal{M} \tag{2.4}$$

for each $k, j \in \mathbb{N}$. We call any left $\mathcal{D}$-module with such a filtration a **filtered module**.

From here, the definition of the Rees algebra follows.

Definition 2.3. For a filtered algebra $\mathcal{D}$, we can define the **Rees algebra** $\mathcal{D}^\hbar$ as:

$$\mathcal{D}^\hbar = \bigoplus_{n\in\mathbb{N}} \hbar^n F_n\mathcal{D} \tag{2.5}$$

which is a graded algebra, graded by $\hbar$.

For the purposes of Airy Ideals, we will need to consider the completion of Rees algebras with respect to the $\hbar$-adic topology:

Definition 2.4. The **completed Rees algebra** $\hat{\mathcal{D}}^\hbar$ is

$$\hat{\mathcal{D}}^\hbar \equiv \prod_{n\in\mathbb{N}} \hbar^n F_n\mathcal{D} \tag{2.6}$$

which is the completion with respect to the $\hbar$-adic topology.

This essentially means that any element $P \in \hat{\mathcal{D}}^\hbar$ can be written as a formal power series in $\hbar$:

$$P = \sum_{n=0}^{\infty} \hbar^n P_n \tag{2.7}$$

for a $P_n \in F_n\mathcal{D}$.

We apply the same construction to $\mathcal{M}$:

Definition 2.5. For a filtered $\mathcal{D}$-module $\mathcal{M}$, we can define the **Rees module** $\mathcal{M}^\hbar$ as:

$$\mathcal{M}^\hbar = \bigoplus_{n\in\mathbb{N}} \hbar^n F_n\mathcal{M} \tag{2.8}$$

which is a graded left $\mathcal{D}$-module, graded by $\hbar$.

Definition 2.6. The **completed Rees module** $\hat{\mathcal{M}}^\hbar$ is

$$\hat{\mathcal{M}}^\hbar \equiv \prod_{n\in\mathbb{N}} \hbar^n F_n\mathcal{M} \tag{2.9}$$

which is the completion with respect to the $\hbar$-adic topology.

2.2 Weyl Algebra

We would like to apply the Rees construction to the Weyl algebra specifically. To define the Weyl algebra, we start with an index set A, that is either finite or countably infinite. We denote $x_A = \{x_a\}_{a\in A}$ as the set of variables x_a for $a \in A$ and $\partial_A = \{\partial_a\}_{a\in A}$ as the set of partial derivatives ∂_a with respect to the variables x_a.

Definition 2.7. If our index set A is countably finite, the **Weyl algebra** is $\mathcal{D}_A = \mathbb{C}[x_A]\langle\partial_A\rangle$ the algebra of differential operators over the polynomial ring $\mathbb{C}[x_A]$, which is in the variables x_A. It is the free associative algebra over $\mathbb{C}$ generated by the set $\{x_A, \partial_A\}$ modulo the commutation relations:

$$[x_a, x_b] = 0, \quad [\partial_a, \partial_b] = 0, \quad [\partial_a, x_b] = \delta_{ab} \tag{2.10}$$

for any $a, b \in A$.

If our index set A is countably infinite, we must define a completion of the Weyl algebra. It should be noted that this completion is unlike the $\hbar$-adhic completion done for the Rees algebra and modules.

Definition 2.8. The **completed Weyl algebra** is the completion of the Weyl algebra that may contain infinite sums of the differential operators, but only polynomial coefficients.

For example, this means that the operator $\sum_{a\in A}\partial_a$ is in $\mathcal{D}_A$, but $\sum_{a\in A} x_A$ is not.

From our construction of the Weyl algebra, there is a very natural exhaustive ascending filtration on $\mathcal{D}_A$. We call this the **Bernstein filtration**. The main idea is that each variable x_a and each partial derivative ∂_a is given degree one, and the subspace $F_i\mathcal{D}_A$ contains all the operators with degree $\leq i$. A more precise definition is given in [4].

From here, we can apply the Rees construction to the Weyl algebra:

Definition 2.9. The **Rees Weyl algebra** $\mathcal{D}^{\hbar}_A$ associated to $\mathcal{D}_A$ with the Bernstein filtration is:

$$\mathcal{D}^{\hbar}_A = \bigoplus_{n\in\mathbb{N}} \hbar^n F_n \mathcal{D}_A \tag{2.11}$$

Again, it is important that we work in a completion of the Rees Weyl algebra for Airy ideals. We define the $\hbar$-adic completion as:

Definition 2.10. The **completed Rees Weyl Algebra** $\hat{\mathcal{D}}^{\hbar}_A$ is:

$$\hat{\mathcal{D}}^{\hbar}_A = \prod_{n\in\mathbb{N}} \hbar^n F_n D_A \tag{2.12}$$

We can think of elements in this algebra in a similar way as Equation 2.7.

With the completed Rees Weyl algebra, it is important to note we limit our study to **bounded** collection of operators $\{P_i\}_{i\in I} \subseteq \hat{\mathcal{D}}^{\hbar}_A$. The technicalities of this is explored in [4], which we gloss over in this paper. The important thing to note is that we work with operators that do not have divergent sums.

2.3 Rees Weyl Modules

Since elements in the Rees Weyl algebra are differential operators, defining polynomials for them to act on is a natural step. This is the motivation for this subsection. We begin with defining the polynomial $\mathcal{D}_a$-module before applying the Rees construction to it.

Definition 2.11. The **polynomial $\mathcal{D}_A$-module** is the polynomial algebra:

$$\mathcal{M}_A = (C)[x_A] \tag{2.13}$$

This module is acted on by the differential operators in $\mathcal{D}_A$ the natural way (i.e. differentiating the polynomials). One can notice that A is possibly infinite, and can contain an infinite sum of differential operators. This is not an issue, as the module only contains polynomials, and so any element of $\mathcal{M}_A$ can only have a finite number of polynomial terms. Thus, even an infinite number of differential operators will collapse an element of $\mathcal{M}_A$ to a finite number of polynomial terms.

From here, we would like to apply the Rees construction to $\mathcal{M}_A$. First, we need a filtration on the module:

Definition 2.12. The **degree filtration** of $\mathcal{M}_A$ is:

$$F_i\mathcal{M}_A = \{\text{polynomials of degree } \leq i\} \tag{2.14}$$

This is a very natural way to define a filtration for $\mathcal{M}_A$. With our filtered module, we can define a Rees polynomial module:

Definition 2.13. The **Rees polynomial module** associated to the degree filtration of $\mathcal{M}_A$ is:

$$\mathcal{M}^\hbar = \bigoplus_{n\in\mathbb{N}} \hbar^n F_n\mathcal{M} \tag{2.15}$$

which is a graded left $\mathcal{D}_A^\hbar$-module.

We then define the $\hbar$-adic completion:

Definition 2.14. The **completion of the Rees polynomial module** $\hat{\mathcal{M}}_A^\hbar$ is

$$\hat{\mathcal{M}}_A^\hbar \equiv \prod_{n\in\mathbb{N}} \hbar^n F_n\mathcal{M}_A \tag{2.16}$$

Elements of this module are formal power series in $\hbar$ of the form:

$$f = \sum_{n=0}^{\infty} \hbar^n f^{(n)}(x_A) \tag{2.17}$$

for polynomials $f^{(n)}(x_A)$ in x_A with degree $\leq n$.

Next, we need to introduce the concept of an annihilator.

Definition 2.15. Let $\mathcal{M}$ be a left $\mathcal{D}$-module. The **annihilator** of an element $\nu \in \mathcal{M}$ is the set in $\mathcal{D}$:

$$\text{Ann}_{\mathcal{D}}(\nu) = \{P \in \mathcal{D} \mid P \cdot \nu = 0\} \tag{2.18}$$

The annihilator is a very important concept in the construction of Airy Ideals. In a sense, it gives the defining characteristic of an Airy Ideal. As for each Airy Ideal, there is a unique (up to normalization) partition function that the Airy Ideal annihilates. The concept of the partition function is made rigorous in the sections that follow.

2.4 $\hat{\mathcal{D}}_A^\hbar$-modules of exponential type

We can think of the completed Rees polynomial $\hat{\mathcal{D}}_A^\hbar$-module $\hat{\mathcal{M}}_A^\hbar$ as a cyclic module generated by 1. We can use this fact to consider more general $\hat{\mathcal{D}}_A^\hbar$-modules called $\hat{\mathcal{D}}_A^\hbar$-modules "of exponential type" by defining some generator:

Definition 2.16. Let

$$Z \equiv \exp\left(\sum_{n=0}^{\infty} h^{n-1} q^{(n+1)}(x_A)\right) \tag{2.19}$$

for polynomials $q^{(n+1)}(x_A)$ in the variables x_A of degree $\leq n+1$. The cyclic left $\hat{\mathcal{D}}_A^\hbar$**-module of exponential type** $\hat{\mathcal{M}}_A^\hbar Z$ is defined to be the module generated by Z.

The operators in $\mathcal{D}$ act on elements $\hat{\mathcal{M}}_A^\hbar Z$ in the standard way (i.e. how differential operators act on polynomials and exponentials).

2.5 Transvections

For our last introduction, we define transvections. "Transvections" are a class of automorphisms in the completed Rees Weyl algebra that transform the differential operators $\hbar\partial_a$, but leave the variables $\hbar x_a$ and $\hbar$ invariant. More precisely, we can say the following:

Definition 2.17. A **transvection** $\phi : \hat{\mathcal{D}}_A^\hbar \to \hat{\mathcal{D}}_A^\hbar$ maps $(\hbar, \hbar x_a, \hbar\partial_a \to (\hbar, \hbar x_a, \bar{H}_a)$, where:

$$\bar{H}_a \equiv \hbar\partial_a + \sum_{n=0}^{\infty} \hbar^n \partial_a q^{(n+1}(x_A) \tag{2.20}$$

and $\partial_a q^{(n+1)}(x_A)$ is the derivative of a polynomial $q^{(n+1)}(x_A)$ with degree $\leq n+1$

Remark As the constant terms of $q^{(n+1)}(x_A)$ are not important, we work with the assumption that the polynomials are normalized so that $q^{(n+1)}(0) = 0$.

Having introduced the preliminary mathematics and terminology, we are ready to define an Airy Ideal.

3 Introduction to Airy Ideals

As we noted earlier, the completed Rees Weyl module is generated by 1. We use this to define the canonical ideal.

Definition 3.1. The **canonical ideal** $\mathcal{I}_{can}$ is:

$$\mathcal{I}_{can} \equiv \mathrm{Ann}_{\hat{\mathcal{D}}_A^\hbar}(1) = \left\{\sum_{a\in A} c_a \hbar\partial_a | c_a \in \hat{\mathcal{D}}_A^\hbar\right\} \tag{3.1}$$

Intuitively, these are any operators in $\hat{\mathcal{D}}_A^\hbar$ with a derivative "attached to the end", so that one maps to zero. This ideal can be viewed as the solution to the equation $\mathcal{I} \cdot Z = 0$, for $Z = 1$. In general, we want Z to be of the exponential type, which brings us to Airy Ideals:

Definition 3.2. An **Airy Ideal** $\mathcal{I}$ is a left ideal in $\hat{\mathcal{D}}_A^\hbar$ for which there exists a transvection ϕ such that $\mathcal{I} = \phi(\mathcal{I}_{can})$

It can be shown that for an Airy Ideal, there always exists a unique Z for which $\mathcal{I} \cdot Z = 0$, where Z is of the form [4]:

$$Z = \exp\left(-\sum_{n=0}^{\infty} h^{n-1} q^{(n+1)}(x_A)\right) \tag{3.2}$$

The Z associated to the left ideal $\mathcal{I}$ is called the **partition function of the Airy Ideal**.

There are a few obvious, but important, properties of Airy Ideals:

1. Since $\mathcal{I} = \phi(\mathcal{I}_{can})$, we have that $\mathcal{I}$ is generated by some operators:

$$\bar{H}_a \equiv \hbar\partial_a + \sum_{n=0}^{\infty} \hbar^n \partial_a q^{(n+1}(x_A) \tag{3.3}$$

2. Since the operators satisfy $[\bar{H}_a, \bar{H}_b] = 0$, we have:

$$[\bar{H}_a, \bar{H}_b] \subseteq \hbar^2 \mathcal{I} \tag{3.4}$$

Note that an infinite linear combination of the generators is well-defined, as $\bar{H}_a$ is bounded.

Airy Ideals have some nice properties that one might want to work with. However, it doesn't seem immediately clear when our left ideal is an Airy Ideal. That is to say, it is not easy or immediate to see if one's ideal belongs in the image of a transvection of the canonical ideal. Fortunately, we have a theorem that makes identifying Airy Ideals much easier.

4 Main Theorem of Study

This brings us to the main theorem of this paper. The machinery we built has been in preparation to understand this theorem and its proof.

Theorem 4.1. *Let $\mathcal{I}$ be a left ideal in $\hat{\mathcal{D}}_A^\hbar$. Then, $\mathcal{I}$ is an Airy Ideal iff:*

- *There exists a bounded generating set of operators $\{H_a\}_{a\in A}$ where $H_a = \hbar\partial_a + O(\hbar^2)$.*
- *The ideal satisfies: $[\mathcal{I}, \mathcal{I}] \subseteq \hbar^2\mathcal{I}$.*

Outline of Proof:
It is easy to show that if $\mathcal{I}$ is an Airy Ideal, then it must satisfy the conditions of this theorem. What is left to show is that the other direction holds.

Thus, the proof of this theorem follows from the proof of a few lemmas, this was shown in [4], and we restate the proof here. We begin with a high-level overview of the lemmas:

- First, we show that there exists a set of first-order differential operators $\bar{H}_a \subseteq \mathcal{I}$ of the form:

$$\bar{H}_a = \hbar\partial_a + \sum_{n=2}^{\infty} \hbar^n p_a^n(x_a) \tag{4.1}$$

  - This shows that $\bar{\mathcal{I}} \subseteq \mathcal{I}$, where $\bar{\mathcal{I}}$ is the left ideal generated by $\{\bar{H}_a\}_{a\in A}$.

- Secondly, we show that $\mathcal{I} \subseteq \bar{\mathcal{I}}$, thus proving that $\mathcal{I} = \bar{\mathcal{I}}$ and $\mathcal{I}$ is generated by $\bar{H}_a$.

- Thirdly, we can show that all polynomials in $\mathcal{I}$ are zero, thus $[\bar{H}_a, \bar{H}_b] = 0$, and by Poincaré lemma, we can write:

$$\bar{H}_a = \hbar\partial_a + \sum_{n=1}^{\infty} \hbar^n \partial_a p_a^{n+1}(x_a) \tag{4.2}$$

As $\bar{H}_a$ generate $\mathcal{I}$, we have that $\mathcal{I} = \phi(\mathcal{I}_{\text{can}})$ for some stable transvection ϕ.
With this roadmap in mind, we can begin the proof.

Lemma 4.2. *Given a left Ideal $\mathcal{I} \subseteq \hat{\mathcal{D}}_A^\hbar$ satisfying the conditions of Theorem 4.1, we have that:*

$$\exists \bar{H}_a \subseteq \mathcal{I} : \bar{H}_a = \hbar\partial_a + \sum_{n=2}^{\infty} \hbar^n p_a^n(x_a) \tag{4.3}$$

Proof
First, we show that for any operator $P \in \hat{\mathcal{D}}_A^\hbar$, we have that:

$$P = \sum_{n=0}^{\infty} \hbar^n p^n(x_A) + Q \tag{4.4}$$

where $p^n(x_A)$ are polynomials of degree $\leq n$ and $Q \in \mathcal{I}$.
We begin by writing P as:

$$P = p^{(0,0)} + \hbar \left(p^{(1,1)} + \sum_{b\in A} p_b^{(1,0)}\partial_b \right) + \hbar^2 \left(p^{(2,2)} + \sum_{b\in A} p_b^{(2,1)}\partial_b + \sum_{b\in A} p_b^{(2,0)}\partial_b\partial_c \right) + O(\hbar^3) \tag{4.5}$$

where $p^{(m,k)}$ denotes the m'th polynomial of degree $\leq k$ of the variables x_A.
For each non-polynomial term in P, we want to replace the right most differential operator, $\hbar\partial_a$, with H_a, up to higher orders in $\hbar$. As we are working in the $\hbar$-adic completion of the Rees Weyl Algebra, we can do this recursively for all $\hbar^n$. Upon doing this, we will have written P as a polynomial plus an operator in $\mathcal{I}$, just as in Equation (4.4).
We start in $O(\hbar)$, replacing $\hbar\partial_b$ with $H_b + O(\hbar^2)$:

$$\hbar \left(p^{(1,1)} + \sum_{b\in A} p_b^{(1,0)}\partial_b \right) \to \hbar p^{(1,1)} + \sum_{b\in A} p_b^{(1,0)} H_b + O(\hbar^2) \tag{4.6}$$

This is beginning to look a bit like Equation (4.4), however, we need to examine $O(\hbar^2)$, as now new terms are introduced by our operation on $O(\hbar)$. We begin by examining H_a, which we can write as:

$$H_a = \hbar\partial_a + \hbar^2 \left(g_a^{(2,2)} + \sum_{b\in A} g_{a;b}^{(2,1)}\partial_b + \sum_{b,c\in A} g_{a;bc}^{(2,0)}\partial_b\partial_c \right) + O(\hbar^3) \tag{4.7}$$

where $g_{\dots}^{(2,i)}$ are polynomials of degree $\leq i$. With this in mind, we have for the $O(\hbar^2)$ term in P:

$$-\hbar^2 \left(\sum_{b\in A} p_b^{(1,0)} g_a^{(2,2)} + \sum_{b,c\in A} p_b^{(1,0)} g_{b;c}^{(2,1)}\partial_c + \sum_{b,c,d\in A} p_b^{(1,0)} g_{b;cd}^{(2,0)}\partial_b\partial_c \right) + O(\hbar^3) \tag{4.8}$$

At this point, it's necessary to address some technicalities regarding the boundedness of this expression. Namely, we need to ensure that these sums are not divergent. Additionally, an argument must be made that each term is in $\mathcal{F}_2\mathcal{D}_A$. While we do not address these technicalities in this paper, they are resolved in [4]. What's crucial to understand is that our expression is well-defined.

We can then repeat our transformation on the $O(\hbar^2)$ terms. Similar to the $O(\hbar)$ case, we will be left with a polynomial term, a term in the ideal, and terms on the order of $O(\hbar^3)$. And since the $O(\hbar^2)$ terms are well-defined, we can apply the same arguments to show that the higher order corrections $O(\hbar^3)$ are well-defined. Applying this process recursively, we will have shown that:

$$P = \sum_{n=0}^{\infty} \hbar^n p^n(x_A) + Q \tag{4.9}$$

From here, the proof of the lemma is simple.

We know that:

$$H_a = \hbar\partial_a + O(\hbar^2) \tag{4.10}$$

which can be written as:

$$H_a = \hbar\partial_a + P \tag{4.11}$$

for $P \in \hat{\mathcal{D}}_A^\hbar$ of $O(\hbar^2)$. From what we have just proven, this can be rewritten as:

$$H_a = \hbar\partial_a + \sum_{n=2}^{\infty} \hbar^n p_a^n(x_A) + Q_a \tag{4.12}$$

Define:

$$\bar{H}_a \equiv H_a - Q_a = \hbar\partial_a + \sum_{n=2}^{\infty} \hbar^n p_a^n(x_A) \tag{4.13}$$

And it is clear that $\bar{H} \in \mathcal{I}$. Doing this for our bounded set of operators $\{H_a\}_{a\in A}$ we get the set of bounded operators $\{\bar{H}_a\}_{a\in A} \subseteq \mathcal{I}$. $\square$

Lemma 4.3. *For a left ideal $\mathcal{I} \subseteq \hat{\mathcal{D}}_A^\hbar$ satisfying the conditions of Theorem 4.1, we have that $\bar{\mathcal{I}} = \mathcal{I}$.*

This means that our left ideal, $\mathcal{I}$, could be thought of as generated from $\{\bar{H}_a\}_{a\in A}$

Proof

By definition, we necessarily have that:

$$\bar{H}_a = H_a - Q_a \tag{4.14}$$

but since $Q_a \in \mathcal{I}$ and $\mathcal{I}$ is generated by H_a, we have that:

$$\bar{H} = H_a - \sum_{b\in A} p_{ab} H_b \tag{4.15}$$

for $p_{ab} \in \hat{\mathcal{D}}_A^\hbar$. And from Lemma 4.2, we know that $p_{ab} = O(\hbar)$. We then note that $H_b = \bar{H}_b + Q_b = \bar{H}_b + \sum_{c\in A} p_{bc} H_c$. Putting this into Equation 4.15, we realize get that:

$$\bar{H}_a = H_a - \sum_{b\in A} p_{ab}\bar{H}_b - \sum_{b,c\in A} p_{ab}p_{bc}H_c \tag{4.16}$$

We have that $p_{ab}p_{bc} = O(\hbar^2)$. Again, there are boundedness technicalities that must be acknowledged here, but we ignore them for the purposes of this paper. In short, recursively performing the steps above is well-defined and yields the equation:

$$\bar{H}_a = H_a - \bar{Q}_a \Longrightarrow H_a = \bar{H}_a + \bar{Q}_a \tag{4.17}$$

where $\bar{Q}_a$ is an infinite sum of linear combinations of $\bar{H}_a$. Furthermore, as we can see, the coefficients start at higher orders of $\hbar$ for each term. Thus, for any finite power of $\hbar$, we only have a finite number of terms, and so $\bar{Q}_a \in \bar{\mathcal{I}}$. Hence, $\mathcal{I} \subseteq \{H_a\} \subseteq \bar{\mathcal{I}} \implies \mathcal{I} = \bar{\mathcal{I}}$ (as we have $\bar{\mathcal{I}} \subseteq \mathcal{I}$ from Lemma 4.2). ☐

Finally, we are able to prove the last lemma required to show the theorem.

Lemma 4.4. *For each $\bar{H}_a$, we can write the operators as:*

$$\bar{H}_a = \hbar\partial_a + \sum_{n=2}^{\infty} \hbar^n \partial_a p_a^{n+1}(x_a) \tag{4.18}$$

Proof

The proof of this last lemma begins with proving that there are no non-zero polynomials in $\mathcal{I}$. We begin with $\bar{H}_a$ written as in Equation 4.13:

$$\bar{H}_a = \hbar\partial_a + \sum_{n=2}^{\infty} \hbar^n p_a^n(x_A) \tag{4.19}$$

In order to show that there are no non-zero polynomials in $\mathcal{I}$, we perform induction on powers of $\hbar$. The proof in [4] utilizes the base case of $N = 2$, but for reasons that will be made apparent later, we use the base case $N = 0$ in this paper. We would like to show that if we assume non-zero polynomials start at $O(\hbar^N)$, then this implies they must start at least at $O(\hbar^{N+1})$, thus, by induction, this means there are no non-zero polynomials. They must all vanish.

Base case $N = 0$:

Since $\mathcal{I}$ is generated by H_a, any polynomial in $\mathcal{I}$ must be a linear combination of H_a. However, it is clear that this is impossible to obtain a polynomial at term h^0 from a linear combination of H_a as defined above.

Inductive Step:

Assume that the claim holds for all $N \in \mathbb{N}$, we need to show that this implies it holds for $N + 1$. First, we begin with the commutator of two operators:

$$[\bar{H}_a, \bar{H}_b] = \left[\hbar\partial_a + \sum_{n=2}^{\infty} \hbar^n p_a^n, \hbar\partial_b + \sum_{n=2}^{\infty} \hbar^n p_b^n\right] = \hbar^2 \sum_{n=2}^{\infty} \hbar^{n-1}(\partial_a p_b^{(n)} - \partial_b p_a^{(n)}) \tag{4.20}$$

Because we have from the condition of the theorem that $[H_a, H_b] \subseteq \hbar^2\mathcal{I}$, this implies:

$$\sum_{n=2}^{\infty} \hbar^{n-1}(\partial_a p_b^{(n)} - \partial_b p_a^{(n)}) \subseteq \mathcal{I} \tag{4.21}$$

Since this is in our ideal, we can invoke the inductive hypothesis, and we know that this must start at least $O(\hbar^N)$. Therefore,

$$\partial_a p_b^{(n)} = \partial_b p_a^{(n)}, \text{ for } a, b \in A \text{ and } 2 \leq n \leq N \tag{4.22}$$

By Poincaré lemma, we can then write:

$$p_a^n = \partial_a q_a^{(n+1)} \text{ for } a \in A \text{ and } 2 \leq n \leq N \tag{4.23}$$

which uniquely fixes the polynomials if we enforce $q^{(n+1)}(0) = 0$. Again, working with an infinite index set A means we need to consider boundedness technicalities for $q^{(n+1)}$. This is elaborated on in [4].

If we taken Equation 4.23 and put it back in Equation 4.19, we get:

$$\bar{H}_a = \hbar\partial_a + \sum_{n=2}^{N} \partial_a q^{(n+1)} + \sum_{n=N+1}^{\infty} \hbar^n p_a^n(x_A) \tag{4.24}$$

Next, we define a stable transvection:

$$\phi_N : (\hbar, \hbar x_a, \hbar\partial_a) \to \left(\hbar, \hbar x_a, \hbar\partial_a - \sum_{n=2}^{N} \hbar^n \partial_a q^{(n+1)}\right) \tag{4.25}$$

We know transvections map the ideal $\mathcal{I}$ to the ideal $\phi_N(\mathcal{I})$, and the generators of $\phi_N(\mathcal{I})$ are:

$$\phi_N(\bar{H}_a) = \hbar\partial_a + \sum_{n=N+1}^{\infty} \hbar^n p_a^n(x_A) \tag{4.26}$$

It is clear that in this ideal, non-zero polynomials must start at $O(\hbar^{N+1})$ (which is done by a similar argument employed in the base case).

We also note that for a polynomial $P \in \mathcal{I}$ we have $\phi_N(P) = P$. And since $\phi_N(P)$ must start at $O(\hbar^{N+1})$, P must start at $O(\hbar^{N+1})$. Thus completing the induction.

Since $[\bar{H}_a, \bar{H}_b] \in \mathcal{I}$ is a polynomial, and all polynomials are zero, we have:

$$[\bar{H}_a, \bar{H}_b] = 0 \tag{4.27}$$

for any arbitrary $a, b \in A$. By Poincaré lemma, we then have:

$$\bar{H}_a = \hbar\partial_a + \sum_{n=2}^{\infty} \partial_a q^{(n+1)} \tag{4.28}$$

for $q^{(n+1)}(0) = 0$. □

Having proved these lemmas, we can proceed with the proof of the main theorem.

Proof of Theorem 4.1

For any left ideal $\mathcal{I}$ satisfying the conditions of the theorem, we have proven it is generated by bounded operators $\{\bar{H}_a\}_{a\in A}$, where $\bar{H}_a$ take the form seen in Equation 4.41. It is then clear that there exists a transvection $\phi : \hat{\mathcal{D}}_A^\hbar \to \hat{\mathcal{D}}_A^\hbar$ that maps $(\hbar, \hbar x_a, \hbar\partial_a) \to (\hbar, \hbar x_a, \bar{H}_a)$ for which $\mathcal{I} = \phi(\mathcal{I}_{can})$ □

4.1 Novel Proofs

In this next section, we introduce a few simple proofs novel to this paper. In particular, we are interested in extending the scope of Theorem 4.1. As it turns out, we are free to add degree one terms, $\hbar p_a^1(x_A)$, to the original operators H_a, and we are still able to maintain the result.

Lemma 4.5. *Let $\mathcal{I}$ be a left ideal in $\hat{\mathcal{D}}_A^\hbar$. Then, $\mathcal{I}$ is an Airy Ideal iff:*

- *There exists a bounded generating set of operators $\{H_a\}_{a\in A}$ where $H_a = \hbar\partial_a + O(\hbar^2) + \hbar p_a^1(x_A)$.*
- *The ideal satisfies: $[\mathcal{I}, \mathcal{I}] \subseteq \hbar^2\mathcal{I}$.*

n.b. the extra term on H_a.

Idea of Proofs

We provide multiple proofs for this lemma. The first proof, while longer, follows a similar approach to the proof of Theorem 4.1, making it a more straightforward extension of the theorem. However, the second proof is considerably more succinct, and is done by explicit construction of the transvection ϕ.

First Proof

The proof of Lemma 4.5 is done in the same manner as the proof of Theorem 4.1, with a few minor changes. The three lemmas we need to show are:

1. There exists a set of first-order differential operators $\bar{H}_a \subseteq \mathcal{I}$ of the form:

$$\bar{H}_a = \hbar\partial_a + \sum_{n=1}^{\infty} \hbar^n p_a^n(x_a) \tag{4.29}$$

   We note that this is different from Equation 4.1, as our summation index begins at $n = 1$, as opposed to $n = 2$.

2. The left ideal generated by these operators is contained within our original left ideal. In essence, $\bar{\mathcal{I}} \subseteq \mathcal{I}$

3. $\left[\bar{H}_a, \bar{H}_b\right] = 0$ for arbitrary $a, b \in A$. This means, by Poincaré lemma, that we can write $\bar{H}_a$ as:

$$\bar{H} = \hbar\partial_a + \sum_{n=1}^{\infty} \partial_a q^{(n+1)} \tag{4.30}$$

   Again, we note that the summation index starts at $n = 1$, as opposed to $n = 2$ for the previous case.

The results of these lemmas are the same as the lemmas used for the proof of Theorem 4.1, however, our summation index of $\bar{H}_a$ starts at $n = 1$, as opposed to $n = 2$. We begin with proof of point 1.

Lemma 4.6. *There exists a set of first-order differential operators $\bar{H}_a \subseteq \mathcal{I}$ of the form:*

$$\bar{H}_a = \hbar\partial_a + \sum_{n=1}^{\infty} \hbar^n p_a^n(x_a) \tag{4.31}$$

Proof
We follow closely the proof of Lemma 4.2.
For any operator $P \in \hat{\mathcal{D}}_A^{\hbar}$:

$$P = f^{(0,0)} + \hbar\left(f^{(1,1)} + \sum_{b\in A} f_b^{(1,0)}\partial_b\right) + \hbar^2\left(f^{(2,2)} + \sum_{b\in A} f_b^{(2,1)}\partial_b + \sum_{b\in A} f_b^{(2,0)}\partial_b\partial_c\right) + O(h^3) \tag{4.32}$$

where $f^{(i,j)}$ is a polynomial of degree $\leq j$.
We want to replace the right most differential operator, ∂_a, with H_a, up to higher orders of $\hbar$ for each non-polynomial term in P. We do this recursively for all orders of h, which is well-defined as we are working in the $\hbar$-adic completion of the Rees Weyl Algebra. First, we note that:

$$H_b = \hbar\partial_b + O(\hbar^2) + \hbar {p_b}^{(1)}(x_A) \tag{4.33}$$

for some polynomials ${p_b}^{(1)}$ of degree 1. This implies that:

$$\hbar\partial_b = H_b + O(\hbar^2) - \hbar {p_b}^{(1)}(x_A) \tag{4.34}$$

We then begin to replace the differential operators, starting at $O(\hbar)$:

$$\hbar\left(f^{(1,1)} + \sum_{b\in A} f_b^{(1,0)}\partial_b\right) \to \underbrace{\hbar f^{(1,1)} + \sum_{b\in A} f_b^{(1,0)} H_b + O(\hbar^2)}_{\text{Same terms as in Lemma 4.2 – repeat steps.}} \underbrace{-\sum_{b\in A} \hbar f_b^{(1,0)} {p_b}^{(1)}}_{\text{Extra term}} \tag{4.35}$$

By the same arguments as in Lemma 4.2, this expression is well-defined, bounded, and is in $\mathcal{F}_2\mathcal{D}_A$. Moving onto order $O(\hbar^2)$, we will also be left with a polynomial term, a term in the ideal, a term on order $O(\hbar^3)$, and an extra term. We find that the extra term is:

$$= -\hbar^2 \sum q^{(2)} \tag{4.36}$$

for some polynomial q of degree at most two. Now applying this process recursively to all orders of $\hbar$, we get:

$$P = \sum_{n=0}^{\infty} \hbar^n f^n(x_A) + Q - \sum_{n=1}^{\infty} \hbar^n q^n(x_A) \tag{4.37}$$

And since we know that, for some P_a:

$$H_a = \hbar\partial_a + P_a \tag{4.38}$$

We can define:

$$\bar{H}_a \equiv H_a - Q_a = \hbar\partial_a + \sum_{n=2}^{\infty} \hbar^n p_a^n(x_A) + \sum_{n=1}^{\infty} \hbar^n q_a^n(x_A) \tag{4.39}$$

$\bar{H}_a$ is clearly in the ideal. And it can be re-written as:

$$\bar{H}_a = \hbar\partial_a + \sum_{n=1}^{\infty} \hbar^n p_a^n(x_A) \tag{4.40}$$

□

The proof that $\mathcal{I} \subseteq \bar{\mathcal{I}}$ remains unchanged from Theorem 4.1.

Additionally, in this paper, we adjusted the proof in Lemma 4.4 so that the inductive step begins at the base case $n = 0$, instead of starting at $n = 2$ as seen in [4]. Thus, this proof holds, and we have no non-zero polynomials in $\mathcal{I}$.

Finally, as $[\bar{H}_a, \bar{H}_b]$ remains a polynomial, Poincaré lemma still applies and we can write $\bar{H}_a$ as:

$$\bar{H}_a = \hbar\partial_a + \sum_{n=1}^{\infty} \partial_a q^{(n+1)} \tag{4.41}$$

It is then clear that there exists a transvection $\phi : \hat{\mathcal{D}}_A^\hbar \to \hat{\mathcal{D}}_A^\hbar$ that maps $(\hbar, \hbar x_a, \hbar\partial_a) \to (\hbar, \hbar x_a, \bar{H}_a)$ for which $\mathcal{I} = \phi(\mathcal{I}_{can})$. □

This concludes our first proof of Lemma 4.5.

Second Proof

Our second proof of the lemma is constructive. We first note that:

$$[H_a, H_b] = \hbar^2(\partial_a p_b^{(1)} - \partial_b p_a^{(1)}) + O(\hbar^3) \tag{4.42}$$

which, by the condition, $[H_a, H_b] \subseteq \hbar^2\mathcal{I}$ implies:

$$(\partial_a p_b^{(1)} - \partial_b p_a^{(1)}) \in \mathcal{I} \Rightarrow \partial_a p_b^{(1)} = \partial_b p_a^{(1)} \tag{4.43}$$

By Poincaré lemma, this implies that for $H_a = \hbar\partial_a + O(\hbar^2) + \hbar p_a^1(x_A)$, we can write $\hbar p_a^1(x_A)$ as $\hbar\partial_a p^2(x_A)$. This allows us to define a transvection:

$$\psi : (\hbar, \hbar x_a, \hbar\partial_a) \to \left(\hbar, \hbar x_a, \hbar\partial_a - \hbar\partial_a p^{(2)}\right) \tag{4.44}$$

Since it preserves the order of $\hbar$, we map $O(\hbar^2)$ to $O(\hbar^2)$, and this transvection will map our generators:

$$\psi(H_a) = \hbar\partial_a + O(\hbar^2) \tag{4.45}$$

Notably, as an automorphism, we also have:

$$\psi([\mathcal{I}, \mathcal{I}]) = [\psi(\mathcal{I}), \psi(\mathcal{I})] \subseteq \psi(\hbar^2\mathcal{I}) \tag{4.46}$$

Therefore, the left ideal defined by the automorphism $\mathcal{I}' \equiv \psi(\mathcal{I})$ satisfies the conditions of Theorem 4.1, and we have an Airy Ideal. More precisely, we have a transvection ϕ from the theorem, and a transvection ψ from construction such that $\mathcal{I} = \psi^{-1}(\phi(\mathcal{I}_{can}))$, where $\psi^{-1} \circ \phi$ is a transvection. □

Remark: Although it seems natural to extend our result of Lemma 4.5 to adding a constant to the generators, this is not possible without further constraints. Using the idea of the second proof, we could define an transvection the same way. However, Equation 4.45 fails to hold, as the $O(\hbar^2)$ terms in H_a are not necessarily mapped to $O(\hbar^2)$ terms.

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